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Anomalous dimensions of fields in a supersymmetric quantum field theory at a renormalization group fixed point

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Abstract. It is shown that any chiral superfield has an anomalous dimension equal to zero when all the couplings of the theory in which it is contained are at a renormalization group fixed point. It is argued that super-QED only has such a fixed point at zero coupling, and thence that the theory is trivial.

1. Introduction

Despite the important implications of the behaviour of the β functions in a quantum theory, very little is known for sure in most field theories away from the origin of coupling constant space. One reason for this is that the zeros at the β function often lie outside the range of normal perturbation theory. Such is the case for the Landau-Ginsburg φ^4 theory associated with the three-dimensional Ising model, where one must use techniques such as the Wilson–Fisher ε-expansion to locate the fixed point.

An exception to this rather disappointing situation is the four-dimensional Wess-Zumino model. It was shown early on in the development of supersymmetry [1] that this theory could only have a zero of the β function at the origin. The argument these authors used was as follows: At a fixed point the theory should exhibit conformal invariance and hence as it is a supersymmetric theory, superconformal invariance. This implies certain superconformal Ward identities for the known conformal three point functions in component field space. After implementing these Ward identities on the form of the three point function in x-space, the authors show that the dimensions of the fields are canonical. Using a previously known theorem they then conclude that the theory must be free at a fixed point.

The purpose of this paper is to extend the above result to other supersymmetric theories. We will begin by giving a simplified version of the FIZ argument for the Wess–Zumino model. This is achieved in two ways, we put their original argument in superspace form, and we also show that the result follows as a simple consequence of the existence of chiral superfields which carry a representation of the superconformal group at the fixed point.

2. Construction of conformal superspace

We begin by giving an argument relating the chiral and dilatation weights of a field, which supplements that given in [2].

The $N=1$ superconformal algebra in four dimensions is generated by the set

$$\{P_\mu, M_{\mu\nu}, D, K_\mu, A, Q_A, \bar{Q}_{\dot{A}}, S_A, \bar{S}_{\dot{A}}\} \tag{1}$$

where P_μ generates translations, $M_{\mu\nu}$ Lorentz rotations, D dilatations, A chiral transformations, and $(Q^A/Q_{\dot{A}})$ and $(S^A/S_{\dot{A}})$ are Majorana spinors generating super-translations and special super-translations respectively. These generators satisfy the following Lie brackets which consitute the conformal algebra:

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}\\
&[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu\\
&[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu\\
&[P_\mu, K_\rho] = 2(\eta_{\mu\rho}D - M_{\mu\rho})\\
&[P_\mu, D] = P_\mu\\
&[K_\mu, D] = -K_\mu
\end{aligned} \tag{2}$$

and also the following super-Lie brackets:

$$\begin{aligned}
&[M_{\mu\nu}, Q_A] = \tfrac{1}{2}\sigma_{\mu\nu A}{}^{B}Q_B & &[M_{\mu\nu}, \bar{Q}_{\dot{A}}] = -\tfrac{1}{2}\bar{\sigma}_{\mu\nu\dot{A}}{}^{\dot{B}}\bar{Q}_{\dot{B}}\\
&[M_{\mu\nu}, S_A] = \tfrac{1}{2}\sigma_{\mu\nu A}{}^{B}S_B & &[M_{\mu\nu}, \bar{S}_{\dot{A}}] = -\tfrac{1}{2}\bar{\sigma}_{\mu\nu\dot{A}}{}^{\dot{B}}\bar{S}_{\dot{B}}\\
&[P_\mu, S_A] = -\mathrm{i}\sigma_{\mu A}{}^{\dot{B}}\bar{Q}_{\dot{B}} & &[P_\mu, \bar{S}_{\dot{A}}] = -\mathrm{i}\bar{\sigma}_{\mu\dot{A}}{}^{B}Q_{\dot{B}}\\
&[K_\mu, Q_A] = \mathrm{i}\sigma_{\mu A}{}^{\dot{B}}\bar{S}_{\dot{B}} & &[K_\mu, \bar{Q}_{\dot{A}}] = \mathrm{i}\bar{\sigma}_{\mu\dot{A}}{}^{B}S_B\\
&[D, Q_A] = \tfrac{1}{2}Q_A & &[D, \bar{Q}_{\dot{A}}] = -\tfrac{1}{2}\bar{Q}_{\dot{A}}\\
&[D, S_A] = \tfrac{1}{2}S_A & &[D, \bar{S}_{\dot{A}}] = \tfrac{1}{2}\bar{S}_{\dot{A}}\\
&[A, Q_A] = 3Q_A & &[A, \bar{Q}_{\dot{A}}] = -3\bar{Q}_{\dot{A}}\\
&[A, S_A] = -3S_A & &[A, \bar{S}_{\dot{A}}] = 3\bar{S}_{\dot{A}}
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\{Q_A, \bar{Q}_{\dot{B}}\} = -2\mathrm{i}\sigma^{\mu}_{A\dot{B}}P_\mu \qquad \{S_A, \bar{S}_{\dot{B}}\} = 2\mathrm{i}\sigma^{\mu}_{A\dot{B}}K_\mu\\
&\{S_A, Q_B\} = 2\varepsilon_{AB}D - \sigma^{\mu\nu}_{AB}M_{\mu\nu} + \varepsilon_{AB}A\\
&\{\bar{S}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 2\varepsilon_{\dot{A}\dot{B}}D + \bar{\sigma}^{\mu\nu}_{\dot{A}\dot{B}}M_{\mu\nu} - \varepsilon_{\dot{A}\dot{B}}A.
\end{aligned} \tag{4}$$

Note our conventions here are identical with those in reference [3].

In order to realize manifestly superconformal invariance in our field theories, it will prove convenient to construct them in superspace [4], which we define here as the coset supermanifold

$$\frac{N=1 \text{ superconformal group}}{(M_{\mu\nu}, D, K_\mu, A, S_A, \bar{S}_{\dot{A}})}. \tag{5}$$

This is an eight-dimensional space, globally parametrized by co-ordinates $(x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$, the point with co-ordinates $(x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$ being canonically associated with the coset representative

$$\mathrm{e}^{z^\pi K_\pi} \equiv \exp(x^\mu P_\mu + \theta^A Q_A + \bar{\theta}^{\dot{A}}\bar{Q}_{\dot{A}}). \tag{6}$$

We may now find a representation of the superconformal algebra terms of the Killing vectors which act on fields defined on this space, i.e. for each generator T we seek an operator δ_T such that

$$[T, \varphi(x_\mu, \theta_A, \bar{\theta}_{\dot{A}})] = \delta_T\varphi(x_\mu, \theta_A, \bar{\theta}_{\dot{A}}). \tag{7}$$

The action of a superconformal transformation on a superfield $\varphi(x_\mu, \theta_A, \bar{\theta}_{\dot{A}})$ may be computed as follows: If T is any generator of the superconformal algebra then

$$T\varphi(x, \theta, \bar{\theta}) = \mathrm{e}^{z.K}(\mathrm{e}^{-z.K}T\,\mathrm{e}^{z.K})\varphi(0) \tag{8}$$

(since $\mathrm{e}^{z.K}\varphi(0) \equiv \varphi(x, \theta, \bar{\theta})$).

This may now be evaluated using the Baker–Campbell–Hausdorff formula:

$$\mathrm{e}^{-z.K}T\,\mathrm{e}^{z.K} = \sum_{n=0}^{\infty} \frac{1}{n!}[[\ldots[T, z.K], z.K], \ldots, z.K]\ (n \text{ brackets}). \tag{9}$$

Since no confusion will arise, we may henceforth denote both the abstract generator T and its associated Killing field δ_T simply by T.

Thus, we find the generators are realized as differential operators on superspace as follows: (N.B. $\partial_\mu \equiv \partial/\partial x^\mu$; $\partial_A \equiv \partial/\partial\theta^A$; $\bar{\partial}_{\dot{A}} \equiv \partial/\partial\bar{\theta}^{\dot{A}}$)

$$\begin{aligned}
&P_\mu = \partial_\mu \qquad Q_A = \partial_A + \mathrm{i}\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\partial_\mu \qquad \bar{Q}_{\dot{A}} = \bar{\partial}_{\dot{A}} + \mathrm{i}\sigma^\mu_{A\dot{A}}\theta^A\partial_\mu \\
&D = -x^\mu\partial_\mu - \tfrac{1}{2}\theta^A\partial_A - \tfrac{1}{2}\bar{\theta}^{\dot{A}}\bar{\partial}_{\dot{A}} + \Delta \\
&M_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu + \tfrac{1}{2}\theta^A\sigma_{\mu\nu A}{}^B\partial_B - \tfrac{1}{2}\bar{\theta}^{\dot{A}}\bar{\sigma}_{\mu\nu\dot{A}}{}^{\dot{B}}\bar{\partial}_{\dot{B}} + \Sigma_{\mu\nu} \\
&A = 3\theta^A\partial_A - 3\bar{\theta}^{\dot{A}}\bar{\partial}_{\dot{A}} + \mathscr{A} \\
&S_A = (\mathrm{i}\theta^2\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}} - \theta_A x^\mu + x_\nu\sigma^{\mu\nu}_{AB}\theta^B)\partial_\mu - 2\theta^2\partial_A + (\mathrm{i}x^\mu\sigma_{\mu A}{}^{\dot{B}} - 2\theta_A\bar{\theta}^{\dot{B}})\bar{\partial}_{\dot{B}} \\
&\qquad + \theta^B\sigma^{\mu\nu}_{AB}\Sigma_{\mu\nu} + 2\theta_A\Delta + \theta_A\mathscr{A} + \mathscr{S}_A \\
&\bar{S}_{\dot{A}} = (\mathrm{i}\bar{\theta}^2\sigma^\mu_{A\dot{A}}\theta^A - \bar{\theta}_{\dot{A}}x^\mu - x_\nu\bar{\sigma}^{\mu\nu}_{\dot{A}\dot{B}}\bar{\theta}^{\dot{B}})\partial_\mu - 2\bar{\theta}^2\bar{\partial}_{\dot{A}} + (\mathrm{i}x^\mu\bar{\sigma}_{\mu\dot{A}}{}^B - 2\bar{\theta}_{\dot{A}}\theta^B)\partial_B \\
&\qquad - \bar{\theta}^{\dot{B}}\bar{\sigma}^{\mu\nu}_{\dot{A}\dot{B}}\Sigma_{\mu\nu} + 2\bar{\theta}_{\dot{A}}\Delta - \bar{\theta}_{\dot{A}}\mathscr{A} + \bar{\mathscr{S}}_{\dot{A}} \\
&K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - \theta^2\bar{\theta}^2\partial_\mu + (x_\mu\theta^A - x^\nu\sigma_{\mu\nu B}{}^A\theta^B - \mathrm{i}\theta^2\bar{\sigma}_{\mu\dot{B}}{}^A\bar{\theta}^{\dot{B}})\partial_A \\
&\qquad + (x_\mu\bar{\theta}^{\dot{A}} + x^\nu\bar{\sigma}_{\mu\nu\dot{B}}{}^{\dot{A}}\bar{\theta}^{\dot{B}} - \mathrm{i}\bar{\theta}^2\sigma_{\mu B}{}^{\dot{A}}\theta^B)\bar{\partial}_{\dot{A}} + \theta^A\sigma_{\rho A\dot{B}}\bar{\theta}^{\dot{B}}\varepsilon_\mu{}^{\nu\lambda\rho}\Sigma_{\nu\lambda} \\
&\qquad + \mathrm{i}\theta^A\sigma_{\mu A}{}^{\dot{B}}\bar{\mathscr{S}}_{\dot{B}} + \mathrm{i}\bar{\theta}^{\dot{A}}\bar{\sigma}_{\mu\dot{A}}{}^B\mathscr{S}_B + \mathrm{i}\theta^A\sigma_{\mu A\dot{B}}\bar{\theta}^{\dot{B}}\mathscr{A} - 2x_\mu\Delta - 2x^\nu\Sigma_{\mu\nu} + \kappa_\mu
\end{aligned} \tag{10}$$

where the operators $\{\Delta, \Sigma_{\mu\nu}, \mathscr{A}, \mathscr{S}_A, \bar{\mathscr{S}}_{\dot{A}}, \kappa_\mu\}$ form a representation of the isotropy group $\{D, M_{\mu\nu}, A, S_A, \bar{S}_{\dot{A}}, K_\mu\}$ at the point $(x, \theta, \bar{\theta}) = (0, 0, 0)$, i.e.

$$\begin{aligned}
D\varphi(0) &= \Delta\varphi(0) & S_A\varphi(0) &= \mathscr{S}_A\varphi(0) \\
M_{\mu\nu}\varphi(0) &= \Sigma_{\mu\nu}\varphi(0) & \bar{S}_{\dot{A}}\varphi(0) &= \bar{\mathscr{S}}_{\dot{A}}\varphi(0) \\
A\varphi(0) &= \mathscr{A}\varphi(0) & K_\mu\varphi(0) &= \kappa_\mu\varphi(0).
\end{aligned} \tag{11}$$

Δ is known as the dilatation weight and $\mathscr{A}$ the chiral weight of the field being acted upon. In an irreducible representation, these are scalars, by an application of Schur's lemma. We wish to consider actions constructed from irreducible supermultiplets of component fields, and in the standard way we achieve this by imposing chiral constraints on our superfields. A superfield φ ($\bar{\varphi}$) is said to be chiral (antichiral) if $\bar{D}_{\dot{A}}\varphi = 0$ ($D_A\bar{\varphi} = 0$), where

$$\begin{aligned}
D_A &= \frac{\partial}{\partial\theta^A} - \mathrm{i}\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\partial_\mu \\
\bar{D}_{\dot{A}} &= \frac{\partial}{\partial\bar{\theta}^{\dot{A}}} - \mathrm{i}\sigma^\mu_{A\dot{A}}\theta^A\partial_\mu
\end{aligned} \tag{12}$$

are the usual super-covariant derivatives.

In a superconformal model, the fields transform as

$$\varphi(z) \mapsto \varphi(z') = e^{t \cdot T} \varphi(z)\, e^{-t \cdot T} \tag{13}$$

where T is any generator of the isotropy group and t is a parameter carrying appropriate Lorentz indices.

Consider the action of a left-handed special supersymmetry transformation on a chiral superfield φ. To lowest order in the parameter $\bar{\eta}^{\dot{A}}$ we have

$$\delta\varphi(z) = [\varphi(z), \bar{\eta}^{\dot{A}} \bar{S}_{\dot{A}}]. \tag{14}$$

In order that this transformation be consistent with chirality we require

$$\bar{D}_{\dot{B}} \delta\varphi(z) = 0 \tag{15}$$

which is satisfied provided

$$\{\bar{S}_{\dot{A}}, \bar{D}_{\dot{B}}\} \varphi(z) = 0. \tag{16}$$

A short shows that

$$\{\bar{S}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 2\varepsilon_{\dot{B}\dot{A}} \Delta - \varepsilon_{\dot{B}\dot{A}} \mathscr{A} - \bar{\sigma}^{\mu\nu}_{\dot{A}\dot{B}} \Sigma_{\mu\nu} + 4\bar{\theta}_{\dot{B}} \bar{D}_{\dot{A}} \tag{17}$$

and applying this to a scalar field, we find that $2\Delta = \mathscr{A}$.

By a similar argument, an antichiral field must satisfy

$$\{S_A, D_B\} \bar{\varphi} = 0. \tag{18}$$

Now

$$\{S_A, D_B\} = 2\varepsilon_{BA} \Delta + \varepsilon_{BA} \mathscr{A} + \sigma^{\mu\nu}_{AB} \Sigma_{\mu\nu} - 4\theta_B D_A \tag{19}$$

and thus an antichiral field has $2\Delta = -\mathscr{A}$.

We may also consider superfields with external Lorentz indices. Taking into account Lorentz rotations, we find that a spinor superfield $\psi_{ABC\ldots}$ with $\bar{D}_{\dot{A}} \psi_{ABC\ldots} = 0$ has $2\Delta = \mathscr{A}$, while $\bar{\psi}_{\dot{A}\dot{B}\dot{C}\ldots}$ with $D_A \bar{\psi}_{\dot{A}\dot{B}\dot{C}\ldots} = 0$ has $2\Delta = -\mathscr{A}$.

For spinor fields with both dotted and undotted indices we find that the condition $\{S_A, D_B\}\psi = 0$ is inconsistent, and thus these fields may not appear in a conformally invariant action. We note also that since a vector index may be replaced by a pair of spinor indices, one dotted, one undotted, fields carrying vector indices also may not appear.

We note here that a similar argument applies in the case of a two-dimensional field theory—here the dilatation and chiral weights are replaced by the so-called superweight $\mathscr{H}$ and chiral charge $\mathscr{T}$. The two-dimensional superconformal algebra contains as a subalgebra the super-Möbius algebra, whose structure mirrors that of the full superconformal algebra in four dimensions. Consequently it follows from the above that a chiral field φ has $2\mathscr{H} = \mathscr{T}$ while an antichiral field $\bar{\varphi}$ has $2\mathscr{H} = -\mathscr{T}$, in agreement with the four dimensional case. This relation is also readily apparent from the $N = 2$ superconformal algebra [9].

3. Immediate consequences

Before looking at the consequences of this result for four-dimensional field theories, it will be illustrative to consider the two-dimensional Landau–Ginsburg models with $N = 2$ supersymmetry.

Landau–Ginzburg models are of interest because of the identification between the conformal models obtained from them at a renormalization group fixed point, and the $N=2$ series of minimal superconformal models (see [5] and references therein).

The action for a Landau-Ginsburg model (in (2,2) Euclidean superspace) is given by

$$S_{LG}=-\int \mathrm{d}^2x\,\mathrm{d}^4\theta\,\bar{\varphi}\varphi+\left\{g\int \mathrm{d}^2x\,\mathrm{d}^2\theta\frac{1}{n!}\varphi^n+\text{c.c.}\right\} \tag{20}$$

where $\bar{D}\varphi=0$. Notice that the interaction term is not classically conformally invariant since the coupling g is dimensional. At a quantum mechanical fixed point, the action will be superconformally invariant. The chiral generator τ_0 is given by

$$\tau_0=-\theta\frac{\partial}{\partial\theta}+\bar{\theta}\frac{\partial}{\partial\bar{\theta}}+\mathcal{T}. \tag{21}$$

If we know that a non-trivial infra-red fixed point exists and that the interaction is of the form of equation (20) at this fixed point then invariance of the interaction term requires that the chiral charge of the field φ is $2/n$ and by the above argument its superweight must be $1/n$. We note that the fixed point is outside the range of normal perturbation theory and that there are quantum corrections to the effective potential.

Returning to four dimensions, we now consider the various renormalizable $N=1$ supersymmetric field theories. The crux of the matter is that the chiral weight of a field may be determined by examining the interaction term of the effective action at the fixed point. Firstly, look at the Wess–Zumino model, whose effective actions contains the terms

$$\Gamma_{WZ}=\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\bar{\varphi}\varphi+\left\{\int \mathrm{d}^4x\,\mathrm{d}^2\theta\frac{\lambda}{3!}\varphi^3+\text{h.c.}\right\} \tag{22}$$

where $\bar{D}_{\dot{A}}\varphi=0$. At the fixed point, we have conformal invariance, and in particular invariance under the chiral transformation, whose corresponding Ward identity is given by:

$$\delta_A\Gamma_{WZ}=\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\delta_{\mathcal{A}}\varphi(x,\theta)\frac{\delta\Gamma_{WZ}}{\delta\varphi(x,\theta)}+\text{h.c.}=0. \tag{23}$$

Substituting the explicit forms of Γ_{WZ} and A, we find that the chiral weight $\mathcal{A}$ of φ is 2. From this we deduce the dilatation weight $\Delta=1$, which is of course the canonical value. Hence we have reproduced the result of reference [1]. It is clear that this result is largely a consequence of the $N=1$ non-renormalization theorem [10]. We now give a manifestly supersymmetric version of the argument of reference [1].

4. Three point correlation function

The superfield three point function $\langle\varphi(x_1,\theta_1,\bar{\theta}_1)\varphi(x_2,\theta_2,\bar{\theta}_2)\varphi(x_3,\theta_3,\bar{\theta}_3)\rangle$ is uniquely determined at the fixed point, since it satisfies the superconformal Ward identities

$$\sum_{i=1}^{3}\Xi^i\langle\varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle=0 \tag{24}$$

where Ξ^i are the operators $\{P^i_\mu, M^i_{\mu\nu}, D^i, K^i_\mu, A^i, Q^i_A, \bar{Q}^i_{\dot{A}}, S^i_A, \bar{S}^i_{\dot{A}}\}$, which are found from the previously given expressions by making the replacements $x\mapsto x^i$, $\theta\mapsto\theta^i$, $\bar{\theta}\mapsto\bar{\theta}^i$ and also $\Delta\mapsto\Delta^i$, $\mathcal{A}\mapsto\mathcal{A}^i$, etc.

Due to the chiral nature of φ, the resulting differential equations are most easily solved by moving to a chiral co-ordinate basis; i.e. we make the replacement $x^\mu = y^\mu + \mathrm{i}\theta^A\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}$ so that the condition $\bar{D}_{\dot{A}}\varphi = 0$ becomes $(\partial/\partial\bar{\theta}^{\dot{A}})\varphi = 0$. The θ dependence of a correlator is determined by the total chiral charge

$$\sum_{i=1}^{3} \mathscr{A}^i \tag{25}$$

of the fields it contains. In this case this is just three times the chiral weight of the field φ. Taking this to be arbitrary at this stage, we have three possibilities (excluding the possibility that $\langle\varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle \equiv 0$). These take the form:

$$\langle\varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle = \theta_1^2\theta_2^2\theta_3^2 f(y_1, y_2, y_3) \tag{26}$$

$$\langle\varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \theta_i^2\theta_{jA}\theta_{kB}(\varepsilon^{AB}f_{ijk} + \sigma_{\mu\nu}^{AB} g_{ijk}^{\mu\nu}) \tag{27}$$

$$\langle\varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} \theta_{iA}\theta_{jB}(\varepsilon^{AB}f_{ij} + \sigma_{\mu\nu}^{AB} g_{ij}^{\mu\nu}) \tag{28}$$

where $f, f_{ijk}, g_{ijk}, f_{ij}, g_{ij}$ are functions of (y_1, y_2, y_3) to be determined by scale invariance and supersymmetry. However, in the first case we find that supersymmetry cannot be satisfied, and in the second case, special supersymmetry fails, so we must reject these possibilities. In the case with total chiral charge 6, the unique solution, with suitable normalization, may be expressed in the form

$$\langle\varphi(y_1, \theta_1)\varphi(y_2, \theta_2)\varphi(y_3, \theta_3)\rangle = \frac{\bar{\Theta}^2}{y_{12}^2 y_{13}^2 y_{23}^2} \tag{29}$$

where $\bar{\Theta}^{\dot{B}} = (y_{12}^\mu\theta_3^A + y_{23}^\mu\theta_1^A + y_{31}^\mu\theta_2^A)\sigma_{\mu A}{}^{\dot{B}}$ and $y_{ij} \equiv y_i - y_j$. Notice that $\bar{\Theta}^{\dot{B}}$ is in fact by itself a super-invariant quantity. It is immediate from the above three point function that the dilatation weight of φ is canonical (i.e. $\Delta = 1$). We may also read off the chiral weight of φ, and verify that the rlation $2\Delta = \mathscr{A}$ is indeed satisfied. Expanding φ in component fields, we find this to be in exact agreement with the component three point functions given in [1].

5. Extension to supersymmetric gauge theories

We now extend this result to supersymmetric gauge theories. In the case of an abelian gauge group this is immediate. The action for pure super-QED is given by

$$S_{\mathrm{SQED}} = \frac{1}{64g^2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, W^A W_A \qquad \text{where } \bar{D}_{\dot{A}} W_A = 0. \tag{30}$$

Proceeding as in the Wess–Zumino case, the chiral Ward identity is given by

$$\delta_A\Gamma_{\mathrm{SQED}} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \delta_{\mathscr{A}} W^A \frac{\delta\Gamma_{\mathrm{SQED}}}{\delta W^A(x, \theta)} + \mathrm{h.c.} = 0. \tag{31}$$

where Γ_{SQED} is the quantum effective action, which contains the classical action S_{SQED} given above. Substitution of the relevant expressions leads to the conclusion that the chiral weight of W_A is 3, and hence using our earlier result the mass dimension of W_A is $\frac{3}{2}$, the canonical value for a spinor field. It should also be possible to construct the three point function $\langle W_A W_B W_C\rangle$ from the superconformal Ward identities it satisfies, however the calculations become rather involved.

Extrapolation to the case of super-Yang–Mills is less straightforward. We have the action

$$S_{\rm SYM}=\frac{1}{64g^2C_2(G)}\int d^4x\, d^2\theta\, {\rm tr}(W^AW_A) \qquad \text{where } {\rm tr}(T_iT_j)=C_2(G)\delta_{ij} \tag{32}$$

T_i being the generators of the gauge group in some representation.

The main complication here is that W_A is not chiral, rather we have $\bar{\mathscr{D}}_{\dot{A}}W_A=0$ where $\bar{\mathscr{D}}_{\dot{A}}$ is a gauge-covariant super-covariant derivative. Instead we may consider the composite gauge invariant operator ${\rm tr}(W^AW_A)$ which does satisfy $\bar{D}_{\dot{A}}({\rm tr}(W^AW_A))=0$. At the fixed point, the action shows that this operator has chiral weight 6, and hence dilatation weight 3. Since chiral and dilatation weights tend to add in the case of composite chiral operators, one would like to deduce from this that the dimension of W_A is canonical.

We may now remark upon the existence of renormalization group fixed points in these theories. In the case of the Wess–Zumino model, it was argued in [1], using the massless extension of the Jost–Schroer theorem [6], that the only fixed point occurs when the coupling constant is zero. From this, it was argued in [7] that the renormalized coupling is forced to zero as the renormalization scale is moved to infinity, i.e. that the theory is trivial.

In the case of super-QED, to which we may couple chiral matter fields in the standard way, giving the action

$$S_{\rm SQED}=\frac{1}{64g^2}\int d^4x\, d^2\theta\, W^AW_A+\int d^4x\, d^4\theta\, \bar{\varphi}\, e^{gV}\varphi \tag{33}$$

the original form of the Jost–Schroer theorem is no longer applicable, since the proof of this theorem requires positive definiteness of the Hilbert space of states of the quantum theory, a condition not applicable in a gauge theory. However it seems that this requirement is not strictly necessary, and an extension of the theorem to electrodynamics was proved by Strocchi in [8]. This theorem states that if the two point function of the electromagnetic field strength, $\langle F_{\mu\nu}F_{\rho\sigma}\rangle$ satisfies the condition

$$\partial_x^{\mu}\langle F_{\mu\nu}(x)F_{\rho\sigma}(0)\rangle=0 \tag{34}$$

then the theory is free. Using superconformal Ward identities we may analyse the two point function $\langle W_A(y_1,\theta_1)W_B(y_2,\theta_2)\rangle$. We are once again using chiral variables here, and the technique is identical to that described in section 4 above. Indeed we find that, up to an arbitrary normalization

$$\langle W_A(y_1,\theta_1)W_B(y_2,\theta_2)\rangle=\frac{\varepsilon_{AB}\theta_{12}^2}{y_{12}^4} \tag{35}$$

where $y_{12}=y_1-y_2$ and $\theta_{12}=\theta_1-\theta_2$.

From the above expression we may easily find the component field correlators. The chiral θ expansion of W_A is

$$W_A=\lambda_A-\theta^B\sigma^{\mu\nu}_{AB}F_{\mu\nu}+\theta_AD-\tfrac{1}{2}\theta^2\zeta_A. \tag{36}$$

Hence we have $F_{BA}=D_{(B}W_{A)}|_{\theta=0}$ where $F_{AB}=\frac{1}{4}\sigma^{\mu\nu}_{AB}F_{\mu\nu}$. Taking appropriate covariant derivatives of $\langle W_AW_B\rangle$ we find that

$$\langle F_{AB}(y_1)F_{CD}(y_2)\rangle=\frac{\varepsilon_{AD}\varepsilon_{BC}+\varepsilon_{BD}\varepsilon_{AC}}{y_{12}^4}. \tag{37}$$

Trading the spinor indices for vectorial indices, we find this to be equivalent to

$$\langle F_{\mu\nu}(y_1)F_{\rho\sigma}(y_2)\rangle = \frac{r_{\mu\rho}r_{\nu\sigma} - r_{\mu\sigma}r_{\nu\rho}}{y_{12}^4} \tag{38}$$

where $r_{\mu\rho} = (\eta_{\mu\rho} - [(y_{12})_\mu (y_{12})_\rho]/y_{12}^2)$. This is the well known expression for the two point function in QED, thus Strocchi's result applies and we may deduce that super-QED at a fixed point is trivial.

In the case of the Wess–Zumino model, it was argued in some detail in reference [7], using arguments of renormalization group flow, that the model is trivial, since the effective coupling is driven to zero in the infra-red limit. This is a consequence of the fact that the β function must necessarily be positive for all values of the coupling, and thus there is a one to one correspondence between the effective coupling and the renormalization scale. We have shown that the β function for super-QED has this property also, and hence it would seem likely that this rather general argument can indeed be extended to the case of super-QED.

If we were able to prove a theorem analogous to that of Strocchi for the case of non-abelian gauge fields, then we would also be able to argue for the triviality of the super-Yang–Mills model. However at present we know of no such extension to the theorem.

6. Summary

From interactions of the types considered we may build the most general renormalizable $N=1$ supersymmetric quantum field theory. We have thus established that all (gauge invariant) fields appearing in any such model have zero anomalous dimension at a renormalization group fixed point. It remains an open question, however, as to whether non-trivial fixed points exist in supersymmetric non-abelian gauge theories.

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